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Direct and Inverse Approximation Theorems for Local Trigonometric Bases

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We investigate the approximation of smooth functions by local trigonometric bases. In particular, we are interested in the local behavior of the approximation error in the L^p-norm. We derive direct and inverse approximation theorems that describe the best approximation on an interval by a finite linear combination of basis functions with support in this interval. As a result, we characterize the Besov spaces on an interval as approximation spaces with respect to a local trigonometric basis. These local results are generalized to the approximation on the real line by linear combinations which are locally finite. The proofs are based on the classical inequalities of Jackson and Timan which are applied to local trigonometric bases by the means of folding and unfolding operators. © 2002 Elsevier Science (USA)

Key Words: Local trigonometric bases; Trigonometric approximation; Jackson's inequality; Timan's inequality; Besov spaces; Approximation spaces.

1. INTRODUCTION

The classical theorems of Bernstein and Jackson relate the smoothness of a periodic function to its approximation quality by trigonometric polynomials. They are the first manifestation of the paradigm of approximation theory that good approximation properties are equivalent to smoothness. The theorems of Bernstein and Jackson continue to influence

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the course of approximation theory, see for example [8, 10, 13]. In particular, the recent invention of wavelet bases and related bases for $L^2(\mathbb{R})$ inevitably poses the challenge to understand the corresponding approximation properties.

In this work, we study the approximation problem with respect to local trigonometric bases (see e.g. [1, 3, 6, 9, 22, 23, 25]). These new bases have been developed in the wake of wavelet theory. Local trigonometric bases can be constructed on an arbitrary partition of the real line, and locally they resemble a trigonometric system. Specifically, assume that a partition of $\mathbb{R}$ is given by an increasing sequence $(a_j)_{j\in\mathbb{Z}}$ such that $\lim_{j\to\pm\infty} a_j = \pm\infty$ and $h_j := a_{j+1} - a_j > 0$, and choose $\epsilon_j > 0$, $j \in \mathbb{Z}$, such that $\epsilon_j + \epsilon_{j+1} \leqslant a_{j+1} - a_j$, $j \in \mathbb{Z}$. Then there exist window functions w_j with $\operatorname{supp} w_j \subseteq [a_j - \varepsilon_j, a_{j+1} + \varepsilon_{j+1}]$ and with given smoothness, such that the functions

$$\psi_{jk}(x) := w_j(x)\sqrt{\frac{2}{h_j}}\cos\left((2k+1)\frac{x-a_j}{2h_j}\pi\right), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{N}_0 \tag{1}$$

form a Riesz basis for $L^2(\mathbb{R})$. For more details and references, we refer to Section 3. This construction is extremely flexible and yields a surprising diversity of Riesz bases for $L^2(\mathbb{R})$. They include the Fourier transform of original wavelet bases of Lemarié and Meyer [24] and the Wilson bases of Daubechies *et al.* [12]. Approximation properties of such bases have been investigated in [2, 4].

Wavelet bases are extremely efficient and successful in many applications because they allow us to measure the local, in fact pointwise, smoothness of a function [21]. Other bases, such as Wilson bases and Gabor frames, have sometimes been discarded because they work with windows of fixed size and the maximum possible localization is the size of the window. In part, this deficit can be overcome by using local trigonometric bases since they admit windows of variable size.

Our goal is to show that the local smoothness can indeed be described and characterized by local trigonometric bases, in particular by Wilson bases. For simplicity, we will restrict our attention to local trigonometric bases with the two-overlapping property. These bases are used in most applications of local trigonometric bases, e.g. signal segmentation, and can be handled with only moderate technical difficulties.

We will characterize the Besov regularity $f \in B^{\alpha}_{p,q}([a_r, a_{s+1}])$, where $r < s$, by means of the approximation properties with respect to a local trigonometric basis (1) associated to a partition $\{a_j : j \in \mathbb{Z}\}$. In analogy with periodic functions that are approximated by trigonometric polynomials, we will approximate f locally by windowed trigonometric polynomials. To take into account the non-uniformity of the partition, we have to adjust the local degree of approximation on each interval $[a_j, a_{j+1}]$.

Specifically, for given $n \in \mathbb{N}_0$, set $n_j := \lceil \frac{2nh_j}{\pi} \rceil$, where $\lceil x \rceil$ is the smallest integer which is greater than or equal to $x \in \mathbb{R}$. Now let Ψ_{nrs} be the linear finite-dimensional subspace of $L^2(\mathbb{R})$ defined by

$$\Psi_{nrs} := \left\{ g = \sum_{j=r}^{s} \sum_{k=0}^{n_j - 1} a_{jk} \psi_{jk} : a_{jk} \in \mathbb{C} \right\}. \tag{2}$$

As a special case of our main results (Theorem 6.1), we may now formulate the characterization of the Hölder–Lipschitz spaces $B^{\alpha}_{\infty,\infty}(I)$, $0 < \alpha < 1$, on an interval I. Recall that these are defined by the condition $|f(x+h) - f(x)| \leqslant C|h|^{\alpha}$ for $x, x+h \in I$.

THEOREM 1.1. *Assume that* $w_j \in B^{\alpha}_{\infty,\infty}(\mathbb{R})$, *for* $j = r, \ldots, s$ *and some* α, $0 < \alpha < 1$.

If $f \in B^{\alpha}_{\infty,\infty}([a_r - \varepsilon_r - \gamma, a_{s+1} + \varepsilon_{s+1} + \gamma])$ for some $\gamma > 0$, then

$$\inf_{g \in \Psi_{nrs}} \|f - g\|_{C([a_r + \varepsilon_r, a_{s+1} - \varepsilon_{s+1}])} = \mathcal{O}(n^{-\alpha}).$$

Conversely, if $\inf_{g \in \Psi_{nrs}} \|f - g\|_{C([a_r - \varepsilon_r, a_{s+1} + \varepsilon_{s+1}])} = \mathcal{O}(n^{-\alpha})$, *then*

$$f \in B^{\alpha}_{\infty,\infty}([a_r + \varepsilon, a_{s+1} - \varepsilon]).$$

While this result falls short of characterizing the pointwise smoothness as could have been done with a wavelet basis, it accomplishes the next best goal, namely the characterization of the smoothness on an arbitrary interval with endpoints in (a_j).

The idea for these characterizations is relatively simple. Any f has an expansion of the form

$$f(x) = \sum\nolimits_{j \in \mathbb{Z}} w_j \sqrt{\frac{2}{h_j}} \left(\sum_{k \in \mathbb{N}_0} a_{jk} \cos\left((2k+1) \frac{x - a_j}{2h_j} \pi \right) \right) \tag{3}$$

with respect to the local trigonometric basis $\{\psi_{jk}\}$. If the windows w_j are compactly supported, sufficiently smooth, and satisfy the two-overlapping condition, then the periodic function $f_j(x) = \sum_{k \in \mathbb{N}_0} a_{jk} \cos((2k+1)\frac{x - a_j}{2h_j}\pi)$ captures the local behavior of f on $\operatorname{supp} w_j$. Consequently, the smoothness of f_j and thus the local smoothness of f on $\operatorname{supp} w_j$ can be determined by means of the approximation properties of f_j by trigonometric polynomials. For the technical execution of this idea, we will make use of Wickerhauser's method of unfolding operators [28, 29]. Our results will be in the style of classical approximation theory: we will first prove a Jackson-type theorem (Theorem 4.1) and then an inverse theorem for local trigonometric bases (Theorem 5.1).

Let us mention a few related ideas. The observation that the local Fourier series f_j capture the behavior of f on $[a_j, a_{j+1}]$ has lead to a characterization of L^p by means of Gabor frames [16, 18]. These results came as a surprise because it had been claimed that Gabor-type expansions or Wilson bases are not suitable for the treatment of L^p questions [11, p. 126]. Secondly, let us emphasize that our results concern the *linear* approximation by functions taken from the subspace Ψ_{nrs}. The case of non-linear approximation with local trigonometric bases has also been treated and lead to results of a completely different nature. For instance, in the n-term approximation problem, f is approximated by a linear combination $g = \sum b_{jk}\psi_{jk}$ with at most n non-zero coefficients b_{jk}. The asymptotic behavior of the approximation error then determines a new class of function spaces, the so-called modulation spaces [19]. These are intimately related to the phase-space (or time-frequency) concentration of functions [15, 17], but have not yet appeared in approximation theory.

The paper is organized as follows: Section 2 surveys a few technical properties of moduli of smoothness, Section 3 provides the required facts for the construction of local trigonometric bases and describes the method of unfolding operators. In Section 4, we state and prove a Jackson-type theorem for local trigonometric bases. Theorem 4.1 makes precise the idea outlined above. The inverse approximation theorem for local trigonometric bases is proved in Section 5. In Section 6, we give a complete characterization of the Besov spaces $B^{\alpha}_{p,q}(I)$ as approximation spaces with respect to a local trigonometric bases.

2. MODULI OF SMOOTHNESS AND BESOV SPACES

In the sequel $X^p(I)$ may denote any of the function spaces $L^p(I)$ if $1 \leqslant p < \infty$, or $C(I)$ if $p = \infty$ with an interval $I = [a, b] \subset \mathbb{R}$, $-\infty \leqslant a < b \leqslant \infty$. Analogously, $X^p_{2\pi}$ denotes the space $C_{2\pi}$ of 2π-periodic continuous functions if $p = \infty$, and if $1 \leqslant p < \infty$ the space $L^p_{2\pi}$ of 2π-periodic p-integrable functions. Finally, X^p can be $X^p_{2\pi}$ or $X^p(I)$ for any interval I.

DEFINITION 2.1. The mth differences of a function f are defined by

$$\Delta_h^m f(x) := \Delta_h^{m-1} f(x+h) - \Delta_h^{m-1} f(x) \quad \text{and} \quad \Delta_h^0 f(x) := f(x).$$

The mth order modulus of smoothness of $f \in X^p(I)$, $I = [a, b]$, is defined by

$$\omega^m(X^p(I), f, \delta) := \sup_{h \in (0,\delta]} \|\Delta_h^m f\|_{X^p([a, b-mh])}.$$

For $f \in X^p_{2\pi}$ we define analogously

$$\omega^m(X^p_{2\pi}, f, \delta) := \sup_{h \in (0,\delta]} \|\Delta^m_h f\|_{X^p_{2\pi}}.$$

We first study some properties of the modulus of smoothness under pointwise multiplication. Using the recursive definition of the mth difference, we obtain a Leibniz rule of the form

$$\Delta^m_h(fg)(x) = \sum_{\mu=0}^{m} \binom{m}{\mu} \Delta^\mu_h f(x) \Delta^{m-\mu}_h g(x+\mu h). \tag{4}$$

This implies for $f \in X^p(I)$ and $g \in C(I)$ that

$$\omega^m(X^p(I), fg, \delta) \leqslant \sum_{\mu=0}^{m} \binom{m}{\mu} \omega^\mu(X^p(I), f, \delta)\omega^{m-\mu}(C(I), g, \delta). \tag{5}$$

Definition 2.2. The Besov space $B^{\alpha,m}_{p,q}(I)$, $\alpha > 0$, $m \in \mathbb{N}_0$, is the set of all $f \in L^p(I)$ such that the semi-norm

$$|f|_{B^{\alpha,m}_{p,q}(I)} := \begin{cases} \left(\int_0^\infty \omega^m(X^p(I), f, t)^q t^{-\alpha q - 1}\, \mathrm{d}t \right)^{1/q} & \text{if } 1 \leqslant q < \infty, \\ \sup_{t \in (0,\infty)} t^{-\alpha} \omega^m(X^p(I), f, t) & \text{if } q = \infty \end{cases}$$

is finite. A norm in $B^{\alpha,m}_{p,q}(I)$ is defined by

$$\|f\|_{B^{\alpha,m}_{p,q}(I)} := \|f\|_{X^p(I)} + |f|_{B^{\alpha,m}_{p,q}(I)}.$$

We will need some properties and equivalent norms of Besov spaces, which we list in the following lemma.

Lemma 2.1. (i) *If* $0 \leqslant \alpha < \beta < m$ *and* $1 \leqslant q_1, q_2, p \leqslant \infty$, *then* $B^{\beta,m}_{p,q_1}(I)$ *is continuously embedded in* $B^{\alpha,m}_{p,q_2}(I)$.

(ii) *For any* $0 \leqslant \alpha < m_1 < m_2$ *and any* $1 \leqslant p, q \leqslant \infty$, *the semi-norms* $|\cdot|_{B^{\alpha,m_1}_{p,q}(I)}$ *and* $|\cdot|_{B^{\alpha,m_2}_{p,q}(I)}$ *are equivalent.*

(iii) *For any* $\alpha \geqslant 0$, $m \in \mathbb{Z}$ *and any* $1 \leqslant p, q \leqslant \infty$, *the semi-norm* $|\cdot|_{B^{\alpha,m}_{p,q}(I)}$ *is equivalent to the discretized semi-norm*

$$|f|^*_{B^{\alpha,m}_{p,q}(I)} := \begin{cases} \left(\sum_{k=0}^{\infty} (2^{k\alpha} \omega^m(X^p(I), f, 2^{-k}))^q \right)^{1/q} & if\ 1 \leqslant q < \infty, \\ \sup_{k \in \mathbb{N}_0} 2^{k\alpha} \omega^m(X^p(I), f, 2^{-k}) & if\ q = \infty. \end{cases}$$

For the proof we refer to [14, Sect. 2.10].

Part (ii) of the lemma allows us to define $B^{\alpha}_{p,q}(I) := B^{\alpha,m}_{p,q}(I)$ for $m-1 \leqslant \alpha < m$ without ambiguity. However, the other equivalent norms for $B^{\alpha}_{p,q}(I)$ will be useful later.

LEMMA 2.2. *Let* $\alpha > 0$, $1 \leqslant p, q \leqslant \infty$.

(i) *If* $f \in B^{\alpha}_{\infty,q}(I)$ *and* $g \in B^{\alpha}_{p,q}(I)$, *then* $fg \in B^{\alpha}_{p,q}(I)$ *and*

$$\|fg\|_{B^{\alpha}_{p,q}(I)} \leqslant C\|f\|_{B^{\alpha}_{\infty,q}(I)}\|g\|_{B^{\alpha}_{p,q}(I)}$$

(ii) *If* $f \in B^{\alpha}_{\infty,q}(I)$ *and* $|f(x)| \geqslant \kappa > 0$ *for* $x \in I$, *then* $\frac{1}{f} \in B^{\alpha}_{\infty,q}(I)$ *and* $\|\frac{1}{f}\|_{B^{\alpha}_{\infty,q}(I)}$ *is bounded by a constant which depends only on* κ, α and $\|f\|_{B^{\alpha}_{\infty,q}(I)}$.

Proof. (i) For $m > \alpha \geqslant m-1$, we can choose α_μ, $\mu = 0, \ldots, m$, such that $\alpha_0 = 0$, $\alpha_\mu < \mu$ when $\mu > 0$, and $\alpha_\mu + \alpha_{m-\mu} = \alpha$. Using the discrete semi-norm of $B^{\alpha}_{p,q}(I)$ and (5), we first estimate in the case $q < \infty$ that

$$\begin{aligned}
|fg|^{*}_{B^{\alpha}_{p,q}(I)} &= \left(\sum_{n=0}^{\infty} (2^{\alpha n}\omega^m(X^p(I), fg, 2^{-n}))^q\right)^{1/q} \\
&\leqslant \left(\sum_{n=0}^{\infty}\left(2^{\alpha n}\sum_{\mu=0}^{m}\binom{m}{\mu}\omega^{m-\mu}(C(I), f, 2^{-n})\omega^{\mu}(X^p(I), g, 2^{-n})\right)^q\right)^{1/q} \\
&\leqslant \sum_{\mu=0}^{m}\binom{m}{\mu}\left(\sum_{n=0}^{\infty}(2^{\alpha_{m-\mu}n}\omega^{m-\mu}(C(I), f, 2^{-n})2^{\alpha_\mu n}\right. \\
&\qquad \left.\times\,\omega^{\mu}(X^p(I), g, 2^{-n}))^q\right)^{1/q}.
\end{aligned}$$

If $\mu > 0$, then the inner sum over n is majorized by

$$\begin{aligned}
&\sup_{n\in\mathbb{N}_0} 2^{\alpha_{m-\mu}n}\omega^{m-\mu}(C(I), f, 2^{-n})\left(\sum_{n=0}^{\infty}(2^{\alpha_\mu n}\omega^{\mu}(X^p(I), g, 2^{-n}))^q\right)^{1/q} \\
&\quad = |f|^{*}_{B^{\alpha_{m-\mu}}_{\infty,\infty}(I)}|g|^{*}_{B^{\alpha_\mu}_{p,q}(I)} \leqslant C\|f\|_{B^{\alpha}_{\infty,q}(I)}\|g\|_{B^{\alpha}_{p,q}(I)},
\end{aligned}$$

where we have used the embeddings of Lemma 2.1(ii).

If $\mu = 0$, then $\alpha_m = \alpha$ and we estimate the q-norm by

$$\left(\sum_{n=0}^{\infty}(2^{\alpha n}\omega^m(C(I), f, 2^{-n})\|g\|_{X^p(I)})^q\right)^{1/q} = |f|^{*}_{B^{\alpha}_{\infty,q}(I)}\|g\|_{X^p(I)}.$$

Altogether we have shown that $\|fg\|_{B^{\alpha}_{p,q}(I)} \leqslant C\|f\|_{B^{\alpha}_{\infty,q}(I)}\|g\|_{B^{\alpha}_{p,q}(I)}$. The proof for $p = \infty$ is similar.

(ii) We will show by induction on m that

$$\left\|\frac{1}{f}\right\|_{B^{\alpha,m}_{\infty,q}(I)} \leqslant \frac{C'_m}{\kappa^{2^m}} \|f\|^{2^m-1}_{B^{\alpha,m}_{\infty,q}(I)}. \tag{6}$$

If $m = 1$ and $\alpha < 1$, then the identity

$$\Delta_h\left(\frac{1}{f}\right)(x) = -\frac{\Delta_h f(x)}{f(x)f(x+h)}$$

implies that $\|\frac{1}{f}\|_{B^{\alpha,1}_{\infty,q}(I)} \leqslant \frac{1}{\kappa^2}\|f\|_{B^{\alpha,1}_{\infty,q}(I)}$. For, $m > 1$, we obtain by (4) that

$$\Delta_h^m\left(\frac{1}{f}\right)(x) = \Delta_h^{m-1}\left(\Delta_h\left(\frac{1}{f}\right)\right)(x) = \sum_{\mu=0}^{m-1}\binom{m-1}{\mu}\Delta_h^{m-\mu}f(x+\mu h)\Delta_h^{\mu}g(x),$$

where $g(x) = -\frac{1}{f(x)}\frac{1}{f(x+h)}$. As in (i), we conclude that

$$\begin{aligned}\left\|\frac{1}{f}\right\|_{B^{\alpha,m}_{\infty,q}(I)} &\leqslant C_m\|f\|_{B^{\alpha,m}_{\infty,q}(I)}\|g\|_{B^{\alpha-1,m-1}_{\infty,q}([a,b-h])} \\ &\leqslant C''_m\|f\|_{B^{\alpha,m}_{\infty,q}(I)}\left\|\frac{1}{f}\right\|^2_{B^{\alpha-1,m-1}_{\infty,q}(I)},\end{aligned}$$

where $I = [a,b]$. By induction hypothesis and Lemma 2.1 we obtain (6). ∎

Furthermore, we will need the following lemma to combine local estimates.

LEMMA 2.3. *Let I_j, $j = 1,\ldots,n$, be a collection of finite intervals, $I = \bigcup_{j=1}^n I_j$, and assume that every $x \in I$ belongs to the interior of at most ℓ intervals I_j. Then there exists a constant C_m independent of I_j, ℓ, p and f such that for $m \in \mathbb{N}$*

$$\sum_{j=1}^n \omega^m(L^p(I_j), f, \delta)^p \leqslant C_m^p \ell\omega^m(L^p(I), f, \delta)^p, \qquad 1 \leqslant p < \infty,$$

$$\sup_{j=1,\ldots,n} \omega^m(C(I_j), f, \delta) \leqslant C_m \ell\omega^m(C(I), f, \delta).$$

Proof. This follows from the equivalence of averaged moduli of smoothness with ordinary moduli of smoothness and a simple estimate for averaged moduli of smoothness [14, Chap. 12, p. 373, inequalities (5.16) and (5.17)]. ∎

3. BIORTHOGONAL LOCAL TRIGONOMETRIC BASES

In the following, we will consider biorthogonal local trigonometric bases in the two-overlapping setting of Chui and Shi [6, 7]. We briefly recall the construction of these bases.

Assume that a partition of $\mathbb{R}$ is given by an increasing sequence $(a_j)_{j\in\mathbb{Z}}$ such that $\lim_{j\to\pm\infty} a_j = \pm\infty$ and $h_j := a_{j+1} - a_j > 0$. Given a sequence of "overlap widths" $\epsilon_j > 0$, $j \in \mathbb{Z}$, we set $a_j^+ := a_j + \epsilon_j$ and $a_j^- := a_j - \epsilon_j$. We will assume throughout that $\epsilon_j + \epsilon_{j+1} \leqslant h_j$. Then the intervals (a_j^-, a_j^+), $j \in \mathbb{Z}$, are pairwise disjoint. Next, we choose a sequence of window functions $w_j : \mathbb{R} \to \mathbb{C}$ associated to the partition $\{a_j\}$ and assume that

$$\operatorname{supp} w_j \subset [a_j^-, a_{j+1}^+]. \tag{7}$$

Then $\operatorname{supp} w_j \cap \operatorname{supp} w_r$ has measure zero if $|j - r| > 1$; this is the so-called two-overlapping setting.

Define the trigonometric system C_{jk} and the cosine wavelets ψ_{jk} by

$$C_{jk}(x) := \sqrt{\frac{2}{h_j}} \cos\left((2k+1)\frac{x - a_j}{2h_j}\pi\right)$$

and

$$\psi_{jk}(x) := w_j(x) C_{jk}(x), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{N}_0. \tag{8}$$

Then $\{C_{jk} : k \in \mathbb{N}_0\}$ forms an orthonormal basis of $L^2([a_j, a_{j+1}])$ for each $j \in \mathbb{Z}$, and for a suitable choice of windows $\{w_j\}$ the set $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ is a Riesz basis for $L^2(\mathbb{R})$. The trigonometric functions C_{jk} in definition (8) can be also replaced by

$$S_{jk}(x) := \sqrt{\frac{2}{h_j}} \sin\left((2k+1)\frac{x - a_j}{2h_j}\pi\right), \qquad k \in \mathbb{N}_0$$

or by

$$D_{jk}(x) := \begin{cases} \sqrt{\frac{1}{h_j}}, & \text{if } k = 0;\ j \text{ even}, \\ \sqrt{\frac{2}{h_j}} \cos\left(k\pi\frac{x - a_j}{h_j}\right) & k = 1, 2, \ldots;\ j \text{ even}, \\ \sqrt{\frac{2}{h_j}} \sin\left(k\pi\frac{x - a_j}{h_j}\right) & k = 1, 2, \ldots;\ j \text{ odd} \end{cases}$$

as well as suitable mixtures of them (cf. [1,3,6]). The results in this paper hold for all types of local trigonometric bases, but for simplicity and convenience we shall state and prove them only for local cosine bases. The simple modifications are left to the reader.

The properties of $\{\psi_{jk}\}$ are best investigated by means of the function-valued matrices

$$\boldsymbol{M}_j(x) := \boldsymbol{M}_{j,w}(x) := \begin{pmatrix} w_j(x) & -w_{j-1}(x) \\ w_j(2a_j - x) & w_{j-1}(2a_j - x) \end{pmatrix}$$

(cf. [6, 7]) and by the associated total unfolding operator $\mathcal{U}_w$ (cf. [3, 22]) defined by

$$\mathcal{U}_w f(x) := w_j(x)\, f(x), \qquad x \in [a_j^+, a_{j+1}^-], \quad j \in \mathbb{Z},$$

$$\begin{pmatrix} \mathcal{U}_w f(x) \\ \mathcal{U}_w f(2a_j - x) \end{pmatrix} := \boldsymbol{M}_j(x) \begin{pmatrix} f(x) \\ f(2a_j - x) \end{pmatrix}, \qquad x \in (a_j, a_j^+), \quad j \in \mathbb{Z},$$

$$\mathcal{U}_w f(a_j) := w_j(a_j) f(a_j), \qquad j \in \mathbb{Z}.$$

This definition determines $\mathcal{U}_w f(x)$ uniquely for every $x \in \mathbb{R}$ and we have

$$\mathcal{U}_w(\chi_{[a_j, a_{j+1})} C_{jk}) = \psi_{jk}.$$

Since $\mathcal{U}_w$ maps the orthonormal basis $\{\chi_{[a_j,a_{j+1}]} C_{jk}\}$ of $L^2(\mathbb{R})$ onto $\{\psi_{jk}\}$, the set ψ_{jk} forms a Riesz basis of $L^2(\mathbb{R})$ if and only if the unfolding operator $\mathcal{U}_w$ is bounded and invertible on $L^2(\mathbb{R})$ (cf. [3, Theorem 9.6]).

The Riesz bounds of $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ are $\|\mathcal{U}_w^{-1}\|_{L^2 \to L^2}^{-2}$ and $\|\mathcal{U}_w\|_{L^2 \to L^2}^2$. They can be calculated precisely in terms of the window functions w_j and the matrices $M_j(x)$ (see [6, Theorem 3]). We will use the following fact: if $\{\psi_{jk}\}$ is a Riesz basis for $L^2(\mathbb{R})$, then $\sup_{j \in \mathbb{Z}} \|w_j\|_\infty < \infty$.

Using the biorthogonality condition

$$\delta_{jr}\delta_{k\ell} = \langle \psi_{jk}, \tilde{\psi}_{r\ell} \rangle = \langle \mathcal{U}_w(\chi_{[a_j,a_{j+1})} C_{jk}), \tilde{\psi}_{r\ell} \rangle = \langle \chi_{[a_j,a_{j+1}]} C_{jk}, \mathcal{U}_w^{\star} \tilde{\psi}_{r\ell} \rangle,$$

we find that the dual basis is given by the functions

$$\tilde{\psi}_{jk} = (\mathcal{U}_w^{\star})^{-1}(\chi_{[a_j,a_{j+1}]} C_{jk}) = \tilde{w}_j\, C_{jk}.$$

The explicit form of the dual window functions $\tilde{w}_j$ (see [6, Theorem 2]) is

$$\tilde{w}_j(x) = \begin{cases} \overline{\frac{1}{w_j(x)}} & \text{if } x \in [a_j^+, a_{j+1}^-], \\ \frac{\overline{w_{j-1}(2a_j - x)}}{\overline{\det \boldsymbol{M}_j(x)}} & \text{if } x \in (a_j^-, a_j^+), \\ \frac{\overline{w_{j+1}(2a_{j+1} - x)}}{\overline{\det \boldsymbol{M}_{j+1}(x)}} & \text{if } x \in (a_{j+1}^-, a_{j+1}^+), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Consequently, the inverse unfolding operator is $U_w^{-1} = \mathcal{U}_{\tilde{w}}^{\star}$ and we obtain that

$$\mathcal{U}_w^{-1} f(x) = \frac{1}{w_j(x)} f(x) = \overline{\tilde{w}_j(x)} f(x), \qquad x \in [a_j^+, a_{j+1}^-],$$

$$\begin{pmatrix} \mathcal{U}_w^{-1} f(x) \\ \mathcal{U}_w^{-1} f(2a_j - x) \end{pmatrix} = \boldsymbol{M}_{j,w}^{-1}(x) \begin{pmatrix} f(x) \\ f(2a_j - x) \end{pmatrix}$$
$$= \boldsymbol{M}_{j,\tilde{w}}^{\star}(x) \begin{pmatrix} f(x) \\ f(2a_j - x) \end{pmatrix}, \qquad x \in (a_j, a_j^+),$$

$$\mathcal{U}_w^{-1} f(a_j) = \frac{1}{w_j(a_j)} f(a_j) = \overline{\tilde{w}_j(a_j)} f(a_j), \qquad j \in \mathbb{Z}.$$

Since the dual basis of a Riesz basis is a Riesz basis, we deduce the following fact.

LEMMA 3.1. *If $\{\psi_{jk}\}$ is a Riesz basis for $L^2(\mathbb{R})$, then*

$$\inf_{j\in\mathbb{Z}} \|\tilde{w}_j\|_{C(\mathbb{R})}^{-1} \geqslant \kappa_w > 0 \quad \textit{and} \quad \inf_{j\in\mathbb{Z}} \inf_{x\in[a_j^-, a_j^+]} \det \boldsymbol{M}_j(x) \geqslant \kappa_w > 0$$

for some constant $\kappa_w > 0$.

For the characterization of the local smoothness of functions by means of local trigonometric bases it is important to understand how the smoothness of the w_j's is related to the smoothness of the dual windows $\tilde{w}_j$.

THEOREM 3.1. *Assume that the collection $\{\psi_{jk}\}$ is a Riesz basis for $L^2(\mathbb{R})$. Let $j \in \mathbb{Z}$, $\alpha > 0$, and $1 \leqslant q \leqslant \infty$. If $\|w_r\|_{B^\alpha_{\infty,q}(\mathbb{R})} \leqslant K$, $r \in \{j-1, j, j+1\}$, then $\|\tilde{w}_j\|_{B^\alpha_{\infty,q}(\mathbb{R})} \leqslant \tilde{K}$, where $\tilde{K}$ depends only on K, α and the constant κ_w of* Lemma 3.1.
In particular, if $\sup_{j\in\mathbb{Z}} \|w_j\|_{B^\alpha_{\infty,q}(\mathbb{R})} \leqslant K$, *then* $\sup_{j\in\mathbb{Z}} \|\tilde{w}_j\|_{B^\alpha_{\infty,q}(\mathbb{R})} \leqslant \tilde{K}$.

Proof. Since the windows are continuous, Lemma 3.1 implies that there is a $\delta_0 > 0$ such that $\det \boldsymbol{M}_r(x) \geqslant \frac{\kappa_w}{2}$ for $x \in [a_r^- - \delta_0, a_r^+ + \delta_0]$, $r \in \{j, j+1\}$. Thus, by Lemma 2.2(i) and (ii) we conclude that the functions

$$v_1(x) := \frac{\overline{w_{j-1}(2a_j - x)}}{\det \boldsymbol{M}_j(x)} \quad \text{and} \quad v_2(x) := \frac{\overline{w_{j+1}(2a_{j+1} - x)}}{\det \boldsymbol{M}_{j+1}(x)}, \quad x \in \mathbb{R}$$

are contained in $B^\alpha_{\infty,q}([a_j^- - \delta_0, a_j^+ + \delta_0])$ and $B^\alpha_{\infty,q}([a_{j+1}^- - \delta_0, a_{j+1}^+ + \delta_0])$, respectively, with norms bounded by a constant depending only on K, α and κ_w. Note that support condition (7) and definition (9) imply that

$$v_1(x) = \begin{cases} 0 & \text{if } x \in [a_j^- - \delta_0, a_j^-], \\ \dfrac{1}{\overline{w_j(x)}} & \text{if } x \in [a_j^+, a_j^+ + \delta_0], \end{cases}$$

$$v_2(x) = \begin{cases} \dfrac{1}{\overline{w_j(x)}} & \text{if } x \in [a_{j+1}^- - \delta_0, a_{j+1}^-], \\ 0 & \text{if } x \in [a_{j+1}^+, a_{j+1}^+ + \delta_0]. \end{cases}$$

Furthermore, from Lemma 3.1 and (6) we know that for δ_0 sufficiently small $\|\frac{1}{w_j}\|_{B^\alpha_{\infty,q}([a_j^+ - \delta_0, a_{j+1}^- + \delta_0])}$ is also bounded by a constant which depends only on K, α and κ_w. If $a_{j+1}^- > a_j^+$ Lemma 2.3 implies

$$\begin{aligned} \|\tilde{w}_j\|_{B^\alpha_{\infty,q}(\mathbb{R})} &\leqslant \|v_1\|_{B^\alpha_{\infty,q}([a_j^- - \delta_0, a_j^+ + \delta_0])} + \|v_2\|_{B^\alpha_{\infty,q}([a_{j+1}^- - \delta_0, a_{j+1}^+ + \delta_0])} \\ &\quad + \left\| \frac{1}{w_j} \right\|_{B^\alpha_{\infty,q}([a_j^+, a_{j+1}^-])} \\ &\leqslant \tilde{K}. \end{aligned}$$

If $a_{j+1}^- = a_j^+$, we use that for $x \in [a_j^+ - \delta_0, a_j^+ + \delta_0]$

$$\tilde{w}_j(x) = v_3(x) := v_1(x) + v_2(x) - \frac{1}{\overline{w_j(x)}}, \qquad x \in [a_j^+ - \delta_0, a_j^+ + \delta_0].$$

Again by Lemma 2.3, we obtain

$$\begin{aligned} \|\tilde{w}_j\|_{B^\alpha_{\infty,q}(\mathbb{R})} &\leqslant \|v_1\|_{B^\alpha_{\infty,q}([a_j^- - \delta_0, a_j^+])} + \|v_2\|_{B^\alpha_{\infty,q}([a_j^+, a_{j+1}^+ + \delta_0])} \\ &\quad + \|v_3\|_{B^\alpha_{\infty,q}([a_j^+ - \delta_0, a_j^+ + \delta_0])} \\ &\leqslant \tilde{K}, \end{aligned}$$

and the theorem is proved. ∎

4. A JACKSON-TYPE THEOREM

Next, we want to investigate the local approximation quality of a smooth function by a finite subset of the local trigonometric basis $\{\psi_{jk}\}$. Since $\psi_{jk} = \mathscr{U}_w(\chi_{[a_j,a_{j+1})} C_{jk})$, we have an error estimate of the form

$$\left\| f - \sum_{r,k} \alpha_{rk}\,\psi_{rk} \right\|_{X^p([a_j,a_{j+1}])} \leqslant \|\mathscr{U}_w\| \left\| \mathscr{U}_w^{-1} f - \sum_{j,k} \alpha_{jk} \chi_{[a_j,a_{j+1})} C_{jk} \right\|_{X^p(\mathbb{R})}.$$

In other words, the error of the local approximation of f by a finite linear combination of elements from $\{\psi_{jk}\}$ can be described by the error of the approximation of $\mathscr{U}_w^{-1} f$ by certain trigonometric polynomials in $[a_j, a_{j+1}]$. This situation is familiar from the classical theorems of Jackson and Bernstein which characterize the smoothness of the approximated function by the order of approximation with trigonometric polynomials. Usually, the function $\mathscr{U}_w^{-1} f = \mathscr{U}_{\tilde{w}}^{\bigstar} f$ has discontinuities at the knots a_j and so these results are not applicable directly, but $\mathscr{U}_w^{-1} f|_{(a_j,a_{j+1})}$ has a smooth periodic extension, which we will study first.

Definition 4.1. For $f \in X^p([a_j^-, a_{j+1}^+])$ (or $f \in X^p(\mathbb{R})$), the operator $\mathscr{F}_j$ is defined by

$$\mathscr{F}_j f(x) := \sum_{r\in\mathbb{Z}} (-1)^r \left(\overline{\tilde{w}_j\left(a_j + 2h_j r + \frac{2h_j}{\pi} x\right)} f\left(a_j + 2h_j r + \frac{2h_j}{\pi} x\right) + \overline{\tilde{w}_j\left(a_j + 2h_j r - \frac{2h_j}{\pi} x\right)} f\left(a_j + 2h_j r - \frac{2h_j}{\pi} x\right) \right),$$

where the windows $\tilde{w}_j$ are given by (9).

Since $\tilde{w}_j$ has compact support, the sum in (10) is locally finite and defines a 2π-periodic function. If $f(x) = 0$ for $x \in [a_j^-, a_{j+1}^+]$, then $\mathscr{F}_j f \equiv 0$ by the support properties of $\tilde{w}_j$. Thus, $\mathscr{F}_j$ acts only on functions with support in $[a_j^-, a_{j+1}^+]$. Furthermore, for $x \in (a_j, a_{j+1})$ we have

$$\begin{aligned} \mathscr{F}_j f\left(\pi \frac{x - a_j}{2h_j}\right) &= \mathscr{U}_w^{-1} f(x) \\ &= \overline{\tilde{w}_j(x)} f(x) + \overline{\tilde{w}_j(2a_j - x)} f(2a_j - x) \\ &\quad - \overline{\tilde{w}_j(2a_{j+1} - x)} f(2a_{j+1} - x), \end{aligned} \tag{11}$$

where the second and the third term occur only when $x \in (a_j, a_j^+)$ and $x \in (a_{j+1}^-, a_{j+1})$, respectively. The technical properties of $\mathscr{F}_j$ are listed in the following lemma.

LEMMA 4.1. *If* $\{\psi_{jk} = w_j C_{jk} : j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ *is a Riesz basis for* $L^2(\mathbb{R})$ *consisting of continuous functions, then each* $\mathscr{F}_j$ *has the following properties*:

(a) Parity: $\mathscr{F}_j f(x) = \mathscr{F}_j f(-x) = -\mathscr{F}_j f(\pi - x)$.

(b) Boundedness: $\mathscr{F}_j$ *is bounded from* $X^p([a_j^-, a_{j+1}^+])$ (*or from* $X^p(\mathbb{R})$) *into* $X^p_{2\pi}$, $1 \leqslant p \leqslant \infty$ *and the operator norm can be estimated by*

$$\|\mathscr{F}_j\|_{X^p([a_j^-,a_{j+1}^+])\to X^p_{2\pi}} \leqslant 2\left(\frac{\pi}{h_j}\right)^{1/p} \|\tilde{w}_j\|_{C(\mathbb{R})}.$$

(c) Smoothness: *Let* $f \in X^p([a_j^- - \gamma, a_{j+1}^+ + \gamma])$ *for some small* $0 < \gamma \leqslant \varepsilon_{j+1}$. *If* $\frac{2h_j}{\pi}\delta \leqslant \frac{\gamma}{m}$, *then the mth order modulus of smoothness can be estimated by*

$$\omega^m(X^p_{2\pi}, \mathscr{F}_j f, \delta) \leqslant 3\left(\frac{2\pi}{h_j}\right)^{1/p} \sum_{\mu=0}^{m} \binom{m}{\mu} \omega^{m-\mu}\left(C(\mathbb{R}), \tilde{w}_j, \frac{2h_j}{\pi}\delta\right)$$

$$\times\, \omega^{\mu}\left(X^p([a_j^- - \gamma, a_{j+1}^+ + \gamma]), f, \frac{2h_j}{\pi}\delta\right).$$

(d) Reconstruction formula: *A function* $f \in X^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$ *can be reconstructed from the* $\mathscr{F}_j f$, $j \in \mathbb{Z}$, *by*

$$f(x) = \sum_{j\in\mathbb{Z}} w_j(x)\mathscr{F}_j f\left(\pi \frac{x - a_j}{2h_j}\right). \tag{12}$$

(e) *The image of* $\psi_{rk} = w_r C_{rk}$ *is*

$$\mathscr{F}_j \psi_{rk} = \delta_{jr}\sqrt{\frac{2}{h_j}}\cos((2k+1)\cdot), \qquad j, r \in \mathbb{Z}, \quad k \in \mathbb{N}_0.$$

Proof. Property (a) follows easily from the definition.

(b) The parity properties imply that $\mathscr{F}_j f$ is completely determined by its values on $[0, \frac{\pi}{2}]$. Therefore, we obtain for $1 \leqslant p < \infty$ that

$$\|\mathscr{F}_j f\|^p_{L^p_{2\pi}} = 4\int_0^{\pi/2} |\mathscr{F}_j f(x)|^p\, dx = \frac{2\pi}{h_j}\int_{a_j}^{a_{j+1}} \left|\mathscr{F}_j f\left(\pi\frac{x - a_j}{2h_j}\right)\right|^p dx$$

$$= \frac{2\pi}{h_j}\left(\int_{a_j}^{a_j^+} \left|\overline{\tilde{w}_j(x)} f(x) + \overline{\tilde{w}_j(2a_j - x)} f(2a_j - x)\right|^p dx\right.$$

$$+ \int_{a_j^+}^{a_{j+1}^-} \left|\overline{\tilde{w}_j(x)} f(x)\right|^p dx + \int_{a_{j+1}^-}^{a_{j+1}} |\overline{\tilde{w}_j(x)} f(x)$$

$$\left. - \overline{\tilde{w}_j(2a_{j+1} - x)} f(2a_{j+1} - x)|^p\, dx\right).$$

Since $|a+b|^p \leqslant 2^{p-1}(|a|^p+|b|^p)$, we obtain

$$\|\mathscr{F}_j f\|_{L^p_{2\pi}}^p \leqslant 2^p \frac{\pi}{h_j} \int_{a_j^-}^{a_{j+1}^+} |\overline{\tilde{w}_j(x)} f(x)|^p \, dx$$

and, consequently,

$$\|\mathscr{F}_j f\|_{L^p_{2\pi}}^p \leqslant 2^p \frac{\pi}{h_j} \|\tilde{w}_j\|_{C(\mathbb{R})}^p \, \|f\|_{L^p[a_j^-, a_{j+1}^+]}^p.$$

If $p=\infty$ and $f \in C(\mathbb{R})$, then the continuity of $\tilde{w}_j$ assures that $\mathscr{F}_j f$ is 2π-periodic and continuous. Using the two-overlapping property again, we obtain

$$\|\mathscr{F}_j f\|_{C_{2\pi}} = \sup_{x \in [a_j, a_{j+1}]} \left| \mathscr{F}_j f\left(\pi \frac{x-a_j}{2h_j}\right)\right| \leqslant 2\|\tilde{w}_j\|_{C(\mathbb{R})} \|f\|_{C([a_j^-, a_{j+1}^+])}.$$

(c) Note that, for given $\alpha > 0$, $\Delta_h^m(f(\alpha\cdot))(x) = (\Delta_{\alpha h}^m f)(\alpha x)$. Writing $\eta = \frac{2h_j}{\pi} h$, the mth order difference of $\mathscr{F}_j f$ is given by

$$\Delta_h^m \mathscr{F}_j f(x) = \sum_{r \in \mathbb{Z}} (-1)^r \left(\Delta_\eta^m(\overline{\tilde{w}_j} f)\left(a_j + 2h_j r + \frac{2h_j}{\pi} x\right) + \Delta_\eta^m(\overline{\tilde{w}_j} f)\left(a_j + 2h_j r - \frac{2h_j}{\pi} x\right)\right).$$

Hence, $\Delta_h^m(\mathscr{F}_j f)$ is 2π-periodic and satisfies the same parity conditions as $\mathscr{F}_j f$. If $\eta \leqslant \frac{\gamma}{m}$ and $x \in [a_j, a_{j+1}]$, then $x + m\eta \in [a_j^-, a_{j+1}^+]$ and by (11) we conclude

$$\Delta_h^m \mathscr{F}_j f\left(\pi \frac{x-a_j}{2h_j}\right) = \Delta_\eta^m(f\overline{\tilde{w}_j})(x) + \Delta_\eta^m(f\overline{\tilde{w}_j})(2a_j - x) - \Delta_\eta^m(f\overline{\tilde{w}_j})(2a_{j+1} - x).$$

Hence, for $\frac{2h_j}{\pi}\delta \leqslant \frac{\gamma}{m}$

$$\begin{aligned}
&\omega^m(X_{2\pi}^p, \mathscr{F}_j f, \delta) \\
&= \sup_{h \in (0, (2h_j/\pi)\delta]} \left(\frac{2\pi}{h_j} \int_{a_j}^{a_{j+1}} \left| \Delta_h^m(f\overline{\tilde{w}_j})(x) + \Delta_h^m(f\overline{\tilde{w}_j})(2a_j - x) - \Delta_h^m(f\overline{\tilde{w}_j})(2a_{j+1} - x)\right|^p dx \right)^{1/p} \\
&\leqslant 3\left(\frac{2\pi}{h_j}\right)^{1/p} \sup_{h \in (0,(2h_j/\pi)\delta]} \|\Delta_h^m(f\overline{\tilde{w}_j})\|_{X^p([a_j^- - mh, a_{j+1}^+])} \\
&\leqslant 3\left(\frac{2\pi}{h_j}\right)^{1/p} \omega^m\left(X^p([a_j^- - \gamma, a_{j+1}^+ + \gamma]), f\overline{\tilde{w}_j}, \frac{2h_j}{\pi}\delta\right). \qquad (13)
\end{aligned}$$

Inequality (13) follows from the triangle inequality, the substitutions $x \to 2a_j - x$ and $x \to 2a_{j+1} - x$ in the second and third integral, and the support condition $\operatorname{supp} \Delta_h^m(f\overline{\tilde{w}_j}) \subseteq [a_j^- - mh, a_{j+1}^+]$. The desired estimate now follows from (5).

Note that $\Delta_h^m(f\tilde{w}_j)$ in (13) does not depend on values of f outside the interval $[a_j^-, a_{j+1}^+]$ because $\operatorname{supp} f\overline{\tilde{w}_j} \subseteq [a_j^-, a_{j+1}^+]$.

(d) For $x \in [a_r^+, a_{r+1}^-]$, $r \in \mathbb{Z}$, the identity

$$\sum_{j\in\mathbb{Z}} w_j(x)\mathscr{F}_j f\left(\pi\frac{x-a_j}{2h_j}\right) = w_r\mathscr{F}_r f(x) = w_r(x)\overline{\tilde{w}_r(x)} f(x) = f(x)$$

follows directly from the definition of $\mathscr{F} f_j$ and the definition of the dual window (9). If $x \in [a_r^-, a_r^+]$, $r \in \mathbb{Z}$, we obtain by (9)

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} w_j(x)\mathscr{F}_j f\left(\pi\frac{x-a_j}{2h_j}\right)\\
&\quad = w_r(x)(\overline{\tilde{w}_r(x)}f(x) + \overline{\tilde{w}_r(2a_r - x)}f(2a_r - x))\\
&\qquad + w_{r-1}(x)(\overline{\tilde{w}_{r-1}(x)}f(x) - \overline{\tilde{w}_{r-1}(2a_r - x)}f(2a_r - x))\\
&\quad = f(x)\left(w_r(x)\frac{w_{r-1}(2a_r - x)}{\det \boldsymbol{M}_r(x)} + w_{r-1}(x)\frac{w_r(2a_r - x)}{\det \boldsymbol{M}_r(x)}\right)\\
&\qquad + f(2a_r - x)\left(w_r(x)\frac{w_{r-1}(x)}{\det \boldsymbol{M}_r(2a_r - x)} - w_{r-1}(x)\frac{w_r(x)}{\det \boldsymbol{M}_r(2a_r - x)}\right)\\
&\quad = f(x).
\end{aligned}$$

Since $r \in \mathbb{Z}$ was arbitrary, (12) holds for all $x \in \mathbb{R}$.

(e) Using $C_{jk}(x) = C_{jk}(2a_j - x) = -C_{jk}(2a_{j+1} - x)$, we obtain

$$\begin{aligned}
&\frac{4}{\pi}\int_0^{\pi/2} \mathscr{F}_j\psi_{rk}(x)\cos((2\ell+1)x)\,dx\\
&\quad = \sqrt{\frac{2}{h_j}}\int_{a_j}^{a_{j+1}} (\psi_{rk}(x)\overline{\tilde{w}_j(x)} + \psi_{rk}(x)\overline{\tilde{w}_j(2a_j - x)}\\
&\qquad\qquad - \psi_{rk}(x)\overline{\tilde{w}_j(2a_{j+1} - x)})C_{jk}(x)\,dx\\
&\quad = \sqrt{\frac{2}{h_j}}\int_{a_j^-}^{a_{j+1}^+} \psi_{rk}(x)\overline{\tilde{\psi}_{j\ell}(x)}\,dx = \sqrt{\frac{2}{h_j}}\langle\psi_{rk}, \tilde{\psi}_{j\ell}\rangle = \sqrt{\frac{2}{h_j}}\delta_{jr}\delta_{k\ell}.
\end{aligned}$$

Since $\{\frac{2}{\sqrt{\pi}}\cos((2\ell+1)\cdot)\}$ is an orthonormal basis for $L^2([0, \frac{\pi}{2}])$, assertion (e) follows immediately. ∎

Remark 4.1. Note that according to the reconstruction formula (12), the trigonometric series $\mathscr{F}_j f$ describes the local behavior of f on the interval $[a_j, a_{j+1}]$. The approximation properties of $\mathscr{F}_j f$ will therefore be related to the smoothness of f on $[a_j, a_{j+1}]$. Therefore, (12) provides the precise technical tool to investigate the local smoothness of f in terms of local trigonometric bases.

We are now ready to consider the approximation of smooth functions by elements of the finite-dimensional space Ψ_{nrs} defined as in (2) by $\Psi_{nrs} := \{g = \sum_{j=r}^{s} \sum_{k=0}^{n_j-1} a_{jk}\psi_{jk} : a_{jk} \in \mathbb{C}\}$, where $n_j := \lceil \frac{2nh_j}{\pi} \rceil$. We consider the best X^p-approximation

$$E_n^{\Psi}(X^p(I), f) := \inf_{g\in\Psi_{n,-\infty,\infty}} \|f-g\|_{X^p(I)} = \inf_{g\in\Psi_{nrs}} \|f-g\|_{X^p(I)}$$

on the interval $I \subset [a_r^+, a_{s+1}^-]$, $-\infty \leqslant r \leqslant s \leqslant \infty$. Note that every $g \in \Psi_{nrs}$ has its support in $[a_r^-, a_{s+1}^+]$ and that $g|_{[a_j^+, a_{j+1}^-]}$ is a windowed trigonometric polynomial of degree $2n_j$. Therefore, it is natural to connect the approximation by Ψ_{nrs} with the approximation of periodic functions by trigonometric polynomials. Denote the space of 2π-periodic trigonometric polynomials of degree at most n by

$$T_n := \left\{ P = \sum_{k=-n}^{n} c_k e^{ik\cdot} : c_k \in \mathbb{C} \right\}$$

and define the subspace $T_n^{01} \subset T_{2n}$ by

$$\begin{aligned} T_n^{01} &:= \left\{ P = \sum_{k=0}^{n-1} c_k \cos((2k+1)\cdot) : c_k \in \mathbb{C} \right\} \\ &= \{P \in T_{2n} : P(x) = P(-x) = -P(\pi - x)\}. \end{aligned}$$

We consider the best approximation

$$E_n(X_{2\pi}^p, f) := \inf_{P\in T_n} \|f-P\|_{X_{2\pi}^p} \quad \text{and} \quad E_n^{01}(X_{2\pi}^p, f) := \inf_{P\in T_n^{01}} \|f-P\|_{X_{2\pi}^p}$$

of a function $f \in X_{2\pi}^p$ by a polynomial from T_n and T_n^{01}, respectively.

LEMMA 4.2. *If $f \in C_{2\pi}$ satisfies the parity conditions $f(x) = f(-x) = -f(\pi - x)$, then $E_{2n}(X_{2\pi}^p, f) = E_n^{01}(X_{2\pi}^p, f)$.*

Proof. Define the reflection operators about 0 and $\pi/2$ to be $R_0 f(x) = f(-x)$ and $R_1 f(x) = -f(\pi - x)$. Then $\|R_0 f\|_{X_{2\pi}^p} = \|R_1 f\|_{X_{2\pi}^p} = \|f\|_{X_{2\pi}^p}$.

Now assume that $f = R_0 f = R_1 f$. Let $P \in T_{2n}$ be the *unique* polynomial of best approximation, i.e., $\|f - P\|_{X^p_{2\pi}} = E_{2n}(X^p_{2\pi}, f)$ (cf. [5, Sect. 2.1.]). Then

$$\|f - R_j P\|_{X^p_{2\pi}} = \|R_j f - R_j P\|_{X^p_{2\pi}} = \|f - P\|_{X^p_{2\pi}} = E_{2n}(X^p_{2\pi}, f)$$

for $j = 1, 2$. Since P is unique, we obtain that $P = R_0 P = R_1 P$ or $P \in T^{01}_n$. Consequently, $E^{01}_n(X^p_{2\pi}, f) = \|f - P\|_{X^p_{2\pi}} = E_{2n}(X^p_{2\pi}, f)$. ∎

The combination of Lemmas 4.1 and 4.2 allows us to apply the classical theorems of Jackson and Timan to the problem of approximating functions by Ψ_{nrs}.

THEOREM 4.1. (A Jackson-type inequality). *Assume that the set* $\{\psi_{jk} = w_j C_{jk} : j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ *is a Riesz basis for* $L^2(\mathbb{R})$ *with continuous windows* w_j *and that* $r, s \in \mathbb{Z}$, $r < s$. *Then, the best approximation of* $f \in X^p([a_r^- - \gamma, a_s^+ + \gamma])$ *with respect to* Ψ_{nrs}, *for* $1 \leqslant p \leqslant \infty$, *and* $\frac{m}{2n} \leqslant \gamma \leqslant \varepsilon_j$, *can be estimated by*

$$E^\Psi_n(X^p([a_r^+, a_{s+1}^-]), f) \leqslant K_m \sup_{j=r,\dots,s} \|w_j\|_{C(\mathbb{R})} \times \left(\sum_{\mu=0}^{m} \sup_{j=r,\dots,s} \omega^{m-\mu}\left(C(\mathbb{R}), \tilde{w}_j, \frac{1}{2n}\right) \times \omega^\mu\left(X^p([a_r^- - \gamma, a_{s+1}^+ + \gamma]), f, \frac{1}{2n}\right)\right). \qquad (14)$$

The constant K_m *depends only on* m.

Proof.
Step 1: *Construction of a good approximation of* f *in* Ψ_{nrs}. Let $P_j = \sqrt{\frac{2}{h_j}} \sum_{k=0}^{n_j-1} a_{jk} \cos((2k+1)\cdot) \in T^{01}_{n_j}$ be the polynomial of best approximation for $\mathscr{F}_j f$, $j = r, \dots, s$, and let Φ_j be the scaled error of approximation defined to be

$$\Phi_j(x) = \mathscr{F}_j f\left(\pi \frac{x - a_j}{2h_j}\right) - \sum_{k=0}^{n_j-1} a_{jk} C_{jk}(x).$$

By definition of P_j and Φ_j, we obtain

$$E^{01}_{n_j}(X^p_{2\pi}, \mathscr{F}_j f) = \|\mathscr{F}_j f - P_j\|_{X^p_{2\pi}} = \left(\frac{2\pi}{h_j}\right)^{1/p} \|\Phi_j\|_{X^p([a_j, a_{j+1}])}. \qquad (15)$$

It is now natural to take the function

$$g_0 = \sum_{j=r}^{s} \sum_{k=0}^{n_j-1} a_{jk}\psi_{jk} \in \Psi_{nrs} \tag{16}$$

as the appropriate approximation of f. Since

$$E_n^{\Psi}(X^p([a_r^+, a_{s+1}^-]), f) \leqslant \|f - g_0\|_{X^p([a_r^+, a_{s+1}^-])},$$

it suffices to further estimate the error $\|f - g_0\|$.

Step 2: *Pointwise and L^p-estimates of $f - g_0$.* Using Lemma 4.1(d) and the condition $\operatorname{supp} w_j \subseteq (a_j^-, a_{j+1}^+)$, we express the pointwise error $f(x) - g_0(x)$ as

$$\begin{aligned} f(x) - g_0(x) &= \sum_{j=r}^{s} w_j(x)\chi_{[a_j^-, a_{j+1}^+]}(x)\left(\mathscr{F}_j f\left(\pi\frac{x - a_j}{2h_j}\right) - \sum_{k=0}^{n_j-1} a_{jk}C_{jk}(x)\right) \\ &= \sum_{j=r}^{s} w_j(x)\chi_{[a_j^-, a_{j+1}^+]}(x)\Phi_j(x). \end{aligned} \tag{17}$$

If $1 < p < \infty$, we apply Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ to the right-hand side of (17) and obtain

$$|f(x) - g_0(x)| \leqslant \left(\sum_{j=r}^{s} |w_j(x)|^q\right)^{1/q} \left(\sum_{j=r}^{s} \chi_{[a_j^-, a_{j+1}^+]}(x)|\Phi_j(x)|^p\right)^{1/p}. \tag{18}$$

Since the windows w_j satisfy the two-overlapping condition, the term in (18) that contains only the w_j's can be estimated uniformly by

$$\left(\sum_{j=r}^{s} |w_j(x)|^q\right)^{1/q} \leqslant \sup_{j\in\mathbb{Z}} (\|w_j\|_{C(\mathbb{R})}^q + \|w_{j+1}\|_{C(\mathbb{R})}^q)^{1/q} \leqslant 2^{1/q} \sup_{j\in\mathbb{Z}} \|w_j\|_{C(\mathbb{R})}.$$

The cases $p = 1$ and ∞ are similar.

Taking the X^p-norm leads to

$$\|f - g_0\|_{X^p([a_r^+, a_{s+1}^-])} \leqslant 2^{1/q} \sup_{j\in\mathbb{Z}} \|w_j\|_{C(\mathbb{R})} \left(\sum_{j=r}^{s} \|\Phi_j\|_{X^p([a_j^-, a_{j+1}^+])}^p\right)^{1/p}.$$

Since Φ_j possesses the same parity properties as the C_{jk}'s by Lemma 4.1, we obtain $\|\Phi_j\|_{X^p([a_j^-, a_{j+1}^+])}^p \leqslant 2\|\Phi_j\|_{X^p([a_j, a_{j+1}])}^p$, and combining with (15) we obtain the estimate

$$\|f - g_0\|_{X^p([a_r^+, a_{s+1}^-])} \leqslant 2^{1+1/q} \sup_{j \in \mathbb{Z}} \|w_j\|_{C(\mathbb{R})} \left(\sum_{j=r}^{s} \frac{h_j}{2\pi} E_{n_j}^{01}(X_{2\pi}^p, \mathcal{F}_j f)^p \right)^{1/p}. \quad (19)$$

Step 3: *Application of Jackson's inequality.* Now Jackson's Theorem [5, Theorem 2.2.1] applies to the periodic functions $\mathcal{F}_j f$ and yields the estimate

$$E_{n_j}^{01}(X_{2\pi}^p, \mathcal{F}_j f) \leqslant C_m\, \omega^m(X_{2\pi}^p, \mathcal{F}_j f, (2n_j)^{-1}) \leqslant C_m\, \omega^m\left(X_{2\pi}^p, \mathcal{F}_j f, \frac{\pi}{4h_j n}\right),$$

where we have used once more the definition $n_j = \lceil \frac{2nh_j}{\pi} \rceil$.

Step 4: *Combination of all estimates.* Next, we substitute the estimates for the modulus of smoothness of $\mathcal{F}_j f$ from Lemma 4.1(c) into (19) and use the triangle inequality to arrive at

$$\begin{aligned} \|f - g_0\|_{X^p([a_r^+, a_{s+1}^-])} &\leqslant C_m \cdot 3 \cdot 2 \sup_{j=r,\ldots,s} \|w_j\|_{C(\mathbb{R})} \\ &\quad \times \sum_{\mu=0}^{m} \binom{m}{\mu} \left(\sum_{j=r}^{s} \omega^{m-\mu}\left(C(\mathbb{R}), \tilde{w}_j, \frac{1}{2n}\right)^p \right. \\ &\quad \left. \times\, \omega^{\mu}\left(X^p([a_j^- - \gamma, a_{j+1}^+ + \gamma]), f, \frac{1}{2n}\right)^p \right)^{1/p}. \end{aligned}$$

Note that the factors $\frac{2h_j}{\pi}$ have canceled everywhere so that the increment in ω^μ is always $\frac{1}{2n}$ independent of j. This is the very reason to introduce the local degree of approximation n_j.

To achieve the final estimate, we apply Lemma 2.3 and obtain

$$\begin{aligned} &\left(\sum_{j=r}^{s} \omega^{m-\mu}\left(C(\mathbb{R}), \tilde{w}_j, \frac{1}{2n}\right)^p \omega^{\mu}\left(X^p([a_j^- - \gamma, a_{j+1}^+ + \gamma]), f, \frac{1}{2n}\right)^p \right)^{1/p} \\ &\leqslant \sup_{j=r,\ldots,s} \omega^{m-\mu}\left(C(\mathbb{R}), \tilde{w}_j, \frac{1}{2n}\right) \\ &\quad \times \left(\sum_{j=r}^{s} \omega^{\mu}\left(X^p([a_j^- - \gamma, a_{j+1}^+ + \gamma]), f, \frac{1}{2n}\right)^p \right)^{1/p} \\ &\leqslant C_m \sup_{j=r,\ldots,s} \omega^{m-\mu}\left(C(\mathbb{R}), \tilde{w}_j, \frac{1}{2n}\right) \omega^{\mu}\left(X^p([a_r^- - \gamma, a_{s+1}^+ + \gamma]), f, \frac{1}{2n}\right) \end{aligned}$$

and the main estimate (14) is proved completely. ∎

Remark 4.2. If we take the limits $r \to -\infty$, $s \to \infty$, we obtain a global Jackson-type inequality of the form

$$E_n^{\Psi}(X^p(\mathbb{R}), f) \leqslant K_m \sup_{j \in \mathbb{Z}} \|w_j\|_{C(\mathbb{R})} \left(\sum_{\mu=0}^{m} \sup_{j \in \mathbb{Z}} \omega^{m-\mu}\left(C(\mathbb{R}), \tilde{w}_j, \frac{1}{2n}\right) \times \omega^{\mu}\left(X^p(\mathbb{R}), f, \frac{1}{2n}\right)\right). \tag{20}$$

Here $\Psi_{n,-\infty,\infty}$ consists of functions $g = \sum_{j \in \mathbb{Z}} (\sum_{k=0}^{n_j} a_{jk} C_{jk}) w_j$. Since the sum over j is locally finite and g is essentially a trigonometric polynomial of degree $2n_j$ on $[a_j, a_{j+1}]$, there are no convergence problems in the approximation by $\Psi_{n,-\infty,\infty}$.

For (20) to make sense, we clearly need a uniform estimate for the smoothness of the dual windows. See Theorem 6.2.

5. AN INVERSE APPROXIMATION THEOREM

Next, we prove an inverse inequality for the approximation by functions in the subspace spanned by Ψ_{nrs}.

For the estimate of the mth order modulus of smoothness of a periodic function $f \in X_{2\pi}^p$, we use Timan's inequality [27, Sect. 6.1.1]

$$\omega^m\left(X_{2\pi}^p, f, \frac{1}{n}\right) \leqslant \frac{c_m}{n^m} \sum_{v=0}^{n} (v+1)^{m-1} E_v(X_{2\pi}^p, f). \tag{21}$$

Because of the monotonicity of $E_v(X_{2\pi}^p, f)$, (21) implies immediately the following modification:

$$\omega^m\left(X_{2\pi}^p, f, \frac{1}{\ell n}\right) \leqslant \frac{c_m}{(\ell n)^m} \sum_{v=0}^{n} \sum_{\lambda=0}^{\ell-1} (v\ell + \lambda + 1)^{m-1} E_{v\ell+\lambda}(X_{2\pi}^p, f) \leqslant \frac{c_m}{n^m} \sum_{v=0}^{n} (v+1)^{m-1} E_{v\ell}(X_{2\pi}^p, f). \tag{22}$$

THEOREM 5.1. *Assume that the set* $\{\psi_{jk} = w_j\, C_{jk} : j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ *is a Riesz basis for* $L^2(\mathbb{R})$ *so that*

$$0 < h_{\min} \leqslant h_j, \qquad j \in \mathbb{Z}. \tag{23}$$

Then, the mth order modulus of smoothness of $f \in X^p([a_r^-, a_{s+1}^+])$ for $-\infty \leqslant r \leqslant s \leqslant \infty$ can be estimated by

$$\omega^m\left(X^p([a_r^+, a_{s+1}^-]), f, \frac{1}{\theta n}\right) \leqslant L_m \times \sup_{j=r,\dots,s} \|\tilde{w}_j\|_{C(\mathbb{R})} \sum_{\mu=0}^{m} \left(\sup_{j=r,\dots,s} \omega^{m-\mu}\left(C(\mathbb{R}), w_j, \frac{1}{\theta n}\right) \times \frac{1}{n^\mu} \sum_{v=0}^{n} (v+1)^{\mu-1} E_v^\Psi(X^p([a_r^-, a_{s+1}^+]), f)\right), \tag{24}$$

where $\theta := \frac{\pi}{2h_{\min}} + 1$ and L_m is a constant depending only on m.

Proof.

Step 1: We first relate the best approximation of f by Ψ_{nrs} to the best approximation of the $\mathscr{F}_j f$'s by trigonometric polynomials. Again, we only deal with $1 \leqslant p < \infty$ and leave the modifications for $p = \infty$ to the reader.

Since $\Psi_{n,j-1,j+2}$ is a finite-dimensional subspace for each $j \in \mathbb{Z}$, there exists a function $g_j \in \Psi_{n,j-1,j+2}$ such that

$$\|f - g_j\|_{X^p([a_j^-, a_{j+1}^+])} = \inf_{g \in \Psi_{n,j-1,j+2}} \|f - g\|_{X^p([a_j^-, a_{j+1}^+])} = E_n^\Psi(X^p([a_j^-, a_{j+1}^+], f).$$

Now Lemma 4.1(b) implies that

$$\|\mathscr{F}_j f - \mathscr{F}_j g_j\|_{X^p_{2\pi}} \leqslant 2\left(\frac{\pi}{h_j}\right)^{1/p} \|\tilde{w}_j\|_{C(\mathbb{R})} \|f - g_j\|_{X^p([a_j^-, a_{j+1}^+])} \leqslant 2\left(\frac{\pi}{h_j}\right)^{1/p} \sup_{\mu=r,\dots,s} \|\tilde{w}_\mu\|_{C(\mathbb{R})} E_n^\Psi(X^p([a_j^-, a_{j+1}^+], f).$$

As a consequence of the two-overlapping condition, we have

$$\sum_{j=r}^{s} E_n^\Psi(X^p([a_j^-, a_{j+1}^+]), f)^p \leqslant \inf_{g \in \Psi_{n,-\infty,\infty}} \sum_{j=r}^{s} \|f - g\|^p_{X^p([a_j^-, a_{j+1}^+])} \leqslant 2 E_n^\Psi(X^p([a_r^-, a_{s+1}^+]), f)^p.$$

Set $\ell_j := \lceil \frac{2h_j}{\pi} \rceil$, then $n\ell_j \geqslant n_j$ and thus

$$\sum_{j=r}^{s} \frac{h_j}{\pi} E_{n\ell_j}(X_{2\pi}^p, \mathscr{F}_j f)^p \leqslant \sum_{j=r}^{s} \frac{h_j}{\pi} E_{n_j}(X_{2\pi}^p, \mathscr{F}_j f)^p$$
$$\leqslant \sum_{j=r}^{s} \frac{h_j}{\pi} \|\mathscr{F}_j f - \mathscr{F}_j g_j\|_{X_{2\pi}^p}^p$$
$$\leqslant 2^{p+1} \sup_{j=r,\ldots,s} \|\tilde{w}_j\|_{C(\mathbb{R})}^p E_n^{\Psi}(X^p([a_r^-, a_{s+1}^+]), f)^p. \quad (25)$$

Step 2: *Application of Timan's inequality* (22). Since $\ell_j = \lceil \frac{2h_j}{\pi} \rceil \leqslant \frac{2h_j}{\pi}(1 + \frac{\pi}{2h_{\min}}) = \frac{2h_j\theta}{\pi}$, we have for $\mu > 0$

$$\sum_{j=r}^{s} \frac{h_j}{\pi} \omega^{\mu}\left(X_{2\pi}^p, \mathscr{F}_j f, \frac{\pi}{2h_j\theta n}\right)^p$$
$$\leqslant \sum_{j=r}^{s} \frac{h_j}{\pi} \omega^{\mu}\left(X_{2\pi}^p, \mathscr{F}_j f, \frac{1}{n\ell_j}\right)^p$$
$$\leqslant \frac{c_\mu^p}{n^{\mu p}} \sum_{j=r}^{s} \frac{h_j}{\pi} \left(\sum_{v=0}^{n} (v+1)^{\mu-1} E_{v\ell_j}(X_{2\pi}^p, \mathscr{F}_j f) \right)^p. \quad (26)$$

By triangle inequality and (25), we obtain

$$\left(\sum_{j=r}^{s} \frac{h_j}{\pi} \omega^{\mu}\left(X_{2\pi}^p, \mathscr{F}_j f, \frac{\pi}{2h_j\theta n}\right)^p \right)^{1/p}$$
$$\leqslant \frac{c_\mu}{n^\mu} \sum_{v=0}^{n} (v+1)^{\mu-1} \left(\sum_{j=r}^{s} \frac{h_j}{\pi} E_{v\ell_j}(X_{2\pi}^p, \mathscr{F}_j f)^p \right)^{1/p}$$
$$\leqslant 2^{(p+1)/p} \frac{c_\mu}{n^\mu} \sup_{j=r,\ldots,s} \|\tilde{w}_j\|_{C(\mathbb{R})} \sum_{v=0}^{n} (v+1)^{\mu-1} E_v^{\Psi}(X^p([a_r^-, a_{s+1}^+]), f). \quad (27)$$

Step 3: *Estimate of the modulus of smoothness.* Applying Lemma 4.1(d) and (5), we obtain that

$$\omega^m\left(X^p([a_j^+, a_{j+2}^-]), f, \frac{1}{\theta n}\right)$$
$$= \omega^m\left(X^p([a_j^+, a_{j+2}^-]), \sum_{i=j}^{j+1} w_i \mathscr{F}_i f\left(\pi \frac{\cdot - a_i}{2h_i}\right), \frac{1}{\theta n}\right)$$
$$\leqslant \sum_{i=j}^{j+1} \omega^m\left(X^p([a_i^-, a_{i+1}^+]), w_i \mathscr{F}_i f\left(\pi \frac{\cdot - a_i}{2h_i}\right), \frac{1}{\theta n}\right) \quad (28)$$

$$\leqslant \sum_{i=j}^{j+1} \left(\frac{h_i}{\pi}\right)^{1/p} \left(\sum_{\mu=0}^{m} \binom{m}{\mu} \omega^{m-\mu}\left(C(\mathbb{R}), w_i, \frac{1}{\theta n}\right) \times \omega^{\mu}\left(X_{2\pi}^{p}, \mathscr{F}_i f, \frac{\pi}{2h_i\theta n}\right)\right).$$

Using $(a+b)^p \leqslant 2^{p-1}(a^p + b^p)$ and (28), we conclude that

$$\omega^m\left(X^p([a_r^+, a_{s+1}^-]), f, \frac{1}{\theta n}\right)^p \leqslant \sum_{j=r}^{s-1} \omega^m\left(X^p([a_j^+, a_{j+2}^-]), f, \frac{1}{\theta n}\right)^p$$

$$\leqslant 2^p \sum_{j=r}^{s} \frac{h_i}{\pi}\left(\sum_{\mu=0}^{m} \binom{m}{\mu} \omega^{m-\mu}\left(C(\mathbb{R}), w_i, \frac{1}{\theta n}\right) \times \omega^{\mu}\left(X_{2\pi}^{p}, \mathscr{F}_i f, \frac{\pi}{2h_i\theta n}\right)\right)^p.$$

Applying the triangle inequality to the last expression, we obtain

$$\omega^m\left(X^p([a_r^+, a_{s+1}^-]), f, \frac{1}{\theta n}\right) \leqslant 2 \sum_{\mu=0}^{m} \binom{m}{\mu} \left(\sum_{j=r}^{s} \omega^{m-\mu}\left(C(\mathbb{R}), w_j, \frac{1}{\theta n}\right)^p \frac{h_j}{\pi} \times \omega^{\mu}\left(X_{2\pi}^{p}, \mathscr{F}_j f, \frac{\pi}{2h_j\theta n}\right)^p\right)^{1/p}. \tag{29}$$

Now, the desired inverse inequality (24) follows by substituting (27) into (29). ∎

Remark 5.1. Note that the term

$$\sum_{v=0}^{n} (v+1)^{-1} E_v^{\Psi}(X^p([a_r^-, a_{s+1}^+]), f)$$

corresponding to $\mu = 0$ can be replaced by $\|f\|_{X^p([a_r^-, a_{s+1}^+])}$ because we have $\omega^0(X_{2\pi}^p, \mathscr{F}_j f, \delta) = \|\mathscr{F}_{prj} f\|_{X_{2\pi}^p}$ in inequality (26). However, this does not affect the asymptotic estimates for $\omega^m(X^p([a_r^+, a_{s+1}^-]), f, \delta)$, $\delta \to 0+$.

6. APPROXIMATION AND SMOOTHNESS

We can now use our results to show equivalence of approximation order and smoothness.

DEFINITION 6.1. The approximation space $A^{\alpha}_{p,q}(I)$, $\alpha > 0$, is the set of all $f \in L^p(I)$ such that the semi-norm

$$|f|_{A^{\alpha}_{p,q}(I)} := \begin{cases} \left(\sum_{n=0}^{\infty} (2^{n\alpha} E^{\Psi}_{2^n}(X^p(I), f))^q \right)^{1/q} & \text{if } 1 \leqslant q < \infty, \\ \sup_{n \in \mathbb{N}_0} (2^{n\alpha} E^{\Psi}_{2^n}(X^p(I), f)) & \text{if } q = \infty \end{cases}$$

is finite. A norm in $A^{\alpha}_{p,q}(I)$ is defined by $\|f\|_{A^{\alpha}_{p,q}(I)} := \|f\|_{X^p(I)} + |f|_{A^{\alpha}_{p,q}(I)}$.

We need a discrete version of a well-known Hardy-type inequality with power weights (see also [20, 26, Theorems 326, 327]).

LEMMA 6.1. *Let* $\mu \in \mathbb{N}$, $0 < \alpha < \mu$, $1 \leqslant q \leqslant \infty$. *If* $a_0 \geqslant a_1 \geqslant \cdots \geqslant 0$ *is a non-increasing sequence*, *then*

$$\sum_{n=0}^{\infty} \left(2^{n(\alpha-\mu)} \sum_{v=0}^{2^n} (v+1)^{\mu-1} a_v \right)^q \leqslant C \sum_{n=0}^{\infty} (2^{n\alpha} a_{2^n-1})^q.$$

Proof. Choose $\epsilon > 0$ such that $\alpha + \epsilon < \mu$. Since a_n is non-increasing, we obtain (by splitting the sum into dyadic blocks)

$$\begin{aligned} \sum_{v=0}^{2^n} (v+1)^{\mu-1} a_v &\leqslant C' \sum_{v=0}^{n} 2^{\mu v} a_{2^v-1} \\ &\leqslant C' \left(\sum_{v=0}^{n} 2^{\epsilon v q'} \right)^{1/q'} \left(\sum_{v=0}^{n} 2^{(\mu-\epsilon) v q} a^q_{2^v-1} \right)^{1/q} \\ &\leqslant C'' 2^{\epsilon n} \left(\sum_{v=0}^{n} 2^{(\mu-\epsilon) v q} a^q_{2^v-1} \right)^{1/q}. \end{aligned}$$

After changing the order of summation, the lemma follows from the estimate

$$\begin{aligned} \sum_{n=0}^{\infty} \left(2^{n(\alpha-\mu)} \sum_{v=0}^{2^n} (v+1)^{\mu-1} a_v \right)^q &\leqslant C''^q \sum_{v=0}^{\infty} 2^{(\mu-\epsilon) v q} a^q_{2^v-1} \sum_{n=v}^{\infty} 2^{n(\alpha-\mu+\epsilon) q} \\ &\leqslant C^q \sum_{v=0}^{\infty} (2^{v\alpha} a_{2^v-1})^q. \quad \blacksquare \end{aligned}$$

We first characterize the local Besov regularity in terms of the approximation properties by a local trigonometric basis.

THEOREM 6.1. *Let $r, s \in \mathbb{Z}$, $\alpha > 0$, and $1 \leqslant p, q \leqslant \infty$ be given. Assume that the set $\{\psi_{jk} = w_j\, C_{jk} : j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ is a Riesz basis for $L^2(\mathbb{R})$ such that $w_j \in B^{\alpha}_{\infty,q}(\mathbb{R})$ for $j = r, \ldots, s$.*

(a) *If $f \in B^{\alpha}_{p,q}([a_r^- - \gamma, a_{s+1}^+ + \gamma])$ for some $\gamma > 0$, then $f \in A^{\alpha}_{p,q}([a_r^+, a_{s+1}^-])$.*
(b) *If $f \in A^{\alpha}_{p,q}([a_r^-, a_{s+1}^+])$, then $f \in B^{\alpha}_{p,q}([a_r^+, a_{s+1}^-])$.*

Proof. (a) We use the Jackson-type inequality of Theorem 4.1 to estimate the approximation space norm for $\alpha < m \leqslant \alpha + 1$ and $1 \leqslant p < \infty$ as follows (and leave the obvious modifications in the case $p = \infty$ to the reader):

$$
\begin{aligned}
|f|_{A^{\alpha}_{p,q}([\alpha_r^+, \alpha_{s+1}^-])} &= \left(\sum_{n=0}^{\infty} (2^{n\alpha} E^{\Psi}_{2^n}(X^p(I), f))^q \right)^{1/q} \\
&\leqslant K_m \sup_{j=r,\ldots,s} \|w_j\|_{C(\mathbb{R})} \Bigg(\sum_{n=0}^{\infty} \Bigg(2^{\alpha n} \sum_{\mu=0}^{m} \sup_{j=r,\ldots,s} \omega^{m-\mu}(C(\mathbb{R}), \tilde{w}_j, 2^{-n-1}) \\
&\quad \times \omega^{\mu}(X^p([a_r^- - \gamma, a_{s+1}^+ + \gamma]), f, 2^{-n-1}) \Bigg)^q \Bigg)^{1/q}. \qquad (30)
\end{aligned}
$$

Since $\sup_{j=r,\ldots,s} \omega^m(\tilde{w}_j) \leqslant \sum_{j=r}^{s} \omega^m(\tilde{w}_j)$, we obtain by means of the triangle inequality that

$$
\begin{aligned}
|f|_{A^{\alpha}_{p,q}([\alpha_r^+, \alpha_{s+1}^-])} &\leqslant K_m \sup_{j=r,\ldots,s} \|w_j\|_{C(\mathbb{R})} \\
&\quad \times \sum_{j=r}^{s} \sum_{\mu=0}^{m} \Bigg(\sum_{n=0}^{\infty} (2^{\alpha n} \omega^{m-\mu}(C(\mathbb{R}), \tilde{w}_j, 2^{-n-1}) \\
&\qquad \times \omega^{\mu}(X^p([a_r^- - \gamma, a_{s+1}^+ + \gamma]), f, 2^{-n-1}))^q \Bigg)^{1/q}. \qquad (31)
\end{aligned}
$$

We have already seen in the proof of Lemma 2.2(i) that the q-norms are majorized by $C\|\tilde{w}_j\|_{B^{\alpha}_{\infty,q}([a_r^- - \gamma, a_{s+1}^+ + \gamma])} \|f\|_{B^{\alpha}_{p,q}([a_r^- - \gamma, a_{s+1}^+ + \gamma])}$. Since $\tilde{w}_j \in B^{\alpha}_{\infty,q}(\mathbb{R})$ by Theorem 3.1, we have proved that $f \in A^{\alpha}_{p,q}([\alpha_r^+, \alpha_{s+1}^-])$ and that $\|f\|_{A^{\alpha}_{p,q}([\alpha_r^+, \alpha_{s+1}^-])} \leqslant C\|f\|_{B^{\alpha}_{p,q}([a_r^- - \gamma, a_{s+1}^+ + \gamma])}$ with a constant C depending only on α and the windows w_j.

(b) To estimate the Besov norm of $f \in A^{\alpha}_{p,q}$, we use the inverse estimate of Theorem 5.1. As before we split α into $\alpha = \alpha_\mu + \alpha_{m-\mu}$ with $\alpha_0 = 0$ and $\alpha_\mu < \mu$.

For $1 \leqslant q < \infty$, we define the principal expressions $D_{j\mu}$ by

$$D_{j\mu}^q = \sum_{n=0}^{\infty} \left(2^{n\alpha_{m-\mu}} \omega^{m-\mu}\left(C(\mathbb{R}), w_j, \frac{1}{\theta 2^n} \right) \times 2^{n(\alpha_\mu - \mu)} \sum_{v=0}^{2^n} (v+1)^{\mu-1} E_v^{\Psi}(X^p([\alpha_r^-, \alpha_{s+1}^+]), f) \right)^q.$$

Then,

$$\begin{aligned} |f|_{B_{p,q}^\alpha} &\leqslant C \left(\sum_{n=0}^{\infty} \left(2^{n\alpha} \omega^m \left(X^p([a_r^+, a_{s+1}^-]), f, \frac{1}{\theta 2^n} \right) \right)^q \right)^{1/q} \\ &\leqslant L_m \sup_{j=r,\dots,s} \|\tilde{w}_j\|_{C(\mathbb{R})} \left(\sum_{n=0}^{\infty} 2^{n\alpha q} \sum_{\mu=0}^{m} \left(\sup_{j=r,\dots,s} \omega^{m-\mu}\left(C(\mathbb{R}), w_j, \frac{1}{\theta 2^n} \right) \times \frac{1}{2^{n\mu}} \sum_{v=0}^{2^n} (v+1)^{\mu-1} E_v^{\Psi}(X^p([a_r^-, a_{s+1}^+]), f) \right)^q \right)^{1/q} \\ &\leqslant L_m \sup_{j=r,\dots,s} \|\tilde{w}_j\|_{C(\mathbb{R})} \sum_{\mu=0}^{m} \sum_{j=r}^{s} D_{j\mu}. \end{aligned} \tag{32}$$

In the estimate of $D_{j\mu}$ for $\mu > 0$, we apply Lemma 6.1 to the sequence $a_v = E_v^{\Psi}(X^p([a_r^-, a_{s+1}^+]), f)$ and obtain

$$\begin{aligned} D_{j\mu} &\leqslant \sup_{n \in \mathbb{N}} 2^{n\alpha_{m-\mu}} \omega^{m-\mu}\left(C(\mathbb{R}), w_j, \frac{1}{\theta 2^n} \right) \times \left(\sum_{n=0}^{\infty} \left(2^{n(\alpha_\mu - \mu)} \sum_{v=0}^{2^n} (v+1)^{\mu-1} E_v^{\Psi}(X^p([a_r^-, a_{s+1}^+]), f) \right)^q \right)^{1/q} \\ &\leqslant |w_j|_{B_{\infty,\infty}^{\alpha_{m-\mu}}(\mathbb{R})} \left(\sum_{n=0}^{\infty} (2^{n\alpha_\mu} E_{2^n-1}^{\Psi}(X^p([a_r^-, a_{s+1}^+]), f))^q \right)^{1/q} \\ &\leqslant C \|w_j\|_{B_{\infty,q}^{\alpha}(\mathbb{R})} \|f\|_{A_{p,q}^{\alpha}([a_r^-, a_{s+1}^+])}. \end{aligned}$$

If $\mu = 0$, then

$$\begin{aligned} D_{j0}^q &= \sum_{n=0}^{\infty} \left(2^{n\alpha} \omega^m \left(C(\mathbb{R}), w_j, \frac{1}{\theta n} \right) \|f\|_{X^p([\alpha_r^-, \alpha_{s+1}^+])} \right)^q \\ &\leqslant \|f\|_{X^p([\alpha_r^-, \alpha_{s+1}^+])}^q \|w_j\|_{B_{\infty,q}^{\alpha}(\mathbb{R})}^q. \end{aligned}$$

Summing over j and μ in (32), we obtain the desired result. ∎

Note that these statements are not completely symmetric. The conclusion is valid on the interval $[a_r^+, a_{s+1}^-]$, whereas the assumption is required on a neighborhood of that interval.

Since $B^\alpha_{\infty,\infty}(I), 0<\alpha<1$, consists of all α-Hölder continuous functions on I, Theorem 1.1 of the introduction follows from Theorem 6.1 as a special case.

To obtain a similar result for the real line, we have to modify the above proof. Note that the treatment of $\sup_{j=r,\dots,s} \omega^{m-\mu}(C(\mathbb{R}), \tilde{w}_j, 2^{-n-1})$ in (31) does no longer work if either $r=-\infty$ or $s=\infty$. We therefore impose a slightly stronger condition on the windows w_j.

THEOREM 6.2. *Assume that the set* $\{\psi_{jk} = w_j\, C_{jk} : j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ *is a Riesz basis for* $L^2(\mathbb{R})$ *and that* $\inf_{j\in\mathbb{Z}} h_j = h_{\min} > 0$. *If for some* $\epsilon > 0$, $\sup_{j\in\mathbb{Z}} \|w_j\|_{B^{\alpha+\epsilon}_{\infty,\infty}(\mathbb{R})} = K < \infty$, *then*

$$A^\alpha_{p,q}(\mathbb{R}) = B^\alpha_{p,q}(\mathbb{R}).$$

Proof. We only explain the necessary modifications.

By Theorem 3.1 the dual windows are in $B^{\alpha+\epsilon}_{\infty,\infty}(\mathbb{R})$ and they satisfy $\sup_{j\in\mathbb{Z}} \|\tilde{w}_j\|_{B^{\alpha+\epsilon}_{\infty,\infty}(\mathbb{R})} = \tilde{K}$.

We have to deal with the expression $\sup_{j\in\mathbb{Z}} \omega^{m-\mu}(C(\mathbb{R}), w_j, 2^{-n-1})$ for $\mu = 0,\dots,m$. If $\mu=0$, then $\sup_{j\in\mathbb{Z}} \omega^{m}(C(\mathbb{R}), w_j, 2^{-n-1}) \leqslant \tilde{K} 2^{-(\alpha+\epsilon)n}$. If $\mu>0$, then Marchaud's inequality [27, Sect. 3.3.3.] implies that for $m-1 \leqslant \alpha < m$

$$\sup_{j\in\mathbb{Z}} \omega^{m-\mu}(C(\mathbb{R}), w_j, 2^{-n-1}) \leqslant K_\mu 2^{(\mu-m)n}.$$

Substituting these inequalities into (30), and (32), we derive that $A^\alpha_{p,q}(\mathbb{R}) = B^\alpha_{p,q}(\mathbb{R})$. The estimates in this case are slightly easier and left to the reader. ∎

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